Public Economics (ECON 131)
Section #1: Optimization Review

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1 Economic Motivation

As you learned in Econ 100A/101A, economists generally assume that individuals seek to make choices that will maximize their utility (optimization) given the limited resources they possess (i.e. the individual's budget or the numbers of hours in a day are constraints).

2 Unconstrained Optimization

To find the maximum of a function, look to the derivative (the slope of a line or curve) of that function for guidance. If a well-behaved function has an interior optimum, it will be at a point where the derivative is equal to 0.

1. Write down the function you wish to find the optimum of (e.g. the utility function).

2. Take the derivative of the function with respect to the choice variable.

3. Set the derivative equal to 0 and rearrange terms to get an expression for the choice variable as a function of everything else.

Example: Find the value of x that maximizes $F(x) = 3x - x^2$.

However, we rarely work with utility functions of this form. "Normal goods" have the property that "more is better," so the unconstrained maximum for a normal good's utility function would be an infinite amount. Individuals don't have limitless wealth to buy as much of any good as they desire. Therefore, we are often looking at situations where individuals have to divide limited resources between at least 2 goods, which leads us to constrained optimization.

3 Constrained Optimization

In Econ 131 we will only deal with choices between 2 goods and will generally assume interior solutions. You will not need to worry about second order conditions. However, you should check whether 0 is preferred to the interior solution (i.e. is the utility from consuming 0 of one good and spending all wealth on the other higher than the utility from the optimum you found) when we are interested in whether an individual participates or not. We will explicitly remind you when we are working with topics where that is an issue.

Note: Since we are only going to be dealing with choices among 2 goods, you will be able to turn any multivariable problem into a single variable problem with a little bit of algebra.

3.1 Mathematical tools: Substitution

The general form of these constrained optimization problems is:

$$\max_{x,z} \; U(x,z) \quad \text{s.t.} \quad w = p_x x + p_z z$$

Note: When there are only 2 goods to choose between you can re-write the problem as simply choosing how much of one's wealth is spent on one of the goods.

- Once you know how much is spent on good x, you know that the remainder of the person's budget is spent on z (i.e. $w - p_x x$ is spent on z). Thus, in a two good world, the choice boils down to how much to spend on x.

- By using substitution, you can convert the problem into a single variable equation with the constraint built in, and now you can solve this the same way you solved the unconstrained maximization.

We then have:

$$w = p_x x + p_z z \ \Rightarrow \ z = \frac{w - p_x x}{p_z}$$

And:

$$\max_{x,z} \ U(x,z) \ \text{ s.t. } \ w = p_x x + p_z z \qquad \Leftrightarrow \qquad \max_{x} \ U\left(x, \frac{w - p_x x}{p_z}\right)$$

1. Now take the full derivative of our re-written utility function with respect to x and set it equal to 0.
2. Solve for x.
3. Substitute x into $z = \dfrac{w - p_x x}{p_z}$ to find the optimal z.

3.2 Mathematical tools: Lagrangean optimization

You may also be familiar with Lagrangean optimization from 100A/101A, an alternative method that allows us to embed the budget constraint in the maximization. Using the same utility function and budget constraint as above, we can express the problem as follows:

$$\max_{X,Z,\lambda} \mathcal{L}(x,z,\lambda) = U(x,z) + \lambda[w - p_x x - p_z z], \tag{1}$$

where λ is a new variable which we've added to the problem, called the *Lagrange multiplier* on the constraint. Notice something interesting about the way we've expressed this problem: when the constraint is fulfilled and the individual uses their entire budget, then our whole second term becomes zero (including the multiplier which we've added). Because we've done that, we haven't changed anything about how the problem behaves at the optimum when we add the budget constraint this way. We can see this even more starkly when we take the first order conditions for maximization, which is to take partial derivatives of the Lagrangean with respect to each variable, then set those equal to zero. First order conditions:

$$\partial\mathcal{L}/\partial X = MU_x - \lambda p_x = 0 \tag{2}$$

$$\partial\mathcal{L}/\partial Z = MU_z - \lambda p_z = 0 \tag{3}$$

$$\partial\mathcal{L}/\partial \lambda = w - p_x x - p_z z = 0 \tag{4}$$

Solving equations (2) (3) and (4) gives the solution for optimal x and z.

Equimarginal principle: You can see that equation (4) shows that at an optimum, the individual should be using all of their income. When we combine equation (2) and (3), we get:

$$\frac{MU_x}{p_x} = \frac{MU_z}{p_z} \tag{5}$$

If we rearrange this, we can get another equivalent (and graphical) interpretation of the first order conditions.

$$MRS_{xz} = \frac{p_x}{p_z} = \frac{MU_x}{MU_z} \tag{6}$$

This says that at an (interior) optimum, the slope of the indifference curve should be equal to the slope of the budget constraint. [1]

3.3 Example 1: Homer Simpson, pizza, and beer

Homer has \$20 to spend on beer (b) and pizza slices (s). A beer costs \$2 while a pizza slice costs \$1. His utility is $U(s, b) = s^{1/2}b^{1/2}$. How many pizza slices and beers should Homer choose to maximize his utility?

[1] There are cases in which this will not work, such as: non-differentiable utility functions (e.g. perfect compliments), non-strictly concave indifference curves (e.g. perfect substitutes), and negative tangency. You should be able to recognize the first two immediately, and we'll need to check to make sure that we're always finding demand to be positive.

4 Cobb-Douglas Preferences

Utility functions of the form $U(A,B) = A^\alpha B^\beta$ with $\alpha + \beta = 1$ are called "Cobb-Douglas" preferences. Functions of this form are used all the time in economics because they are very user friendly and have some nice properties.

- **General Form:** $U(A,B) = A^\alpha B^\beta$ with $\alpha + \beta = 1$
- **Example:** $U(A,B) = A^{0.25} B^{0.75}$

4.1 Useful properties of C-D preferences

1. They can be re-written as $U(A,B) = A^\alpha B^{1-\alpha}$
2. They represent two normal goods with positive diminishing marginal utility in each good.

$$\frac{\partial U}{\partial A} = \alpha A^{\alpha-1} B^{1-\alpha} \qquad \frac{\partial^2 U}{\partial A^2} = \alpha(\alpha-1) A^{\alpha-2} B^{1-\alpha}$$

$$\frac{\partial U}{\partial B} = (1-\alpha) A^{\alpha} B^{-\alpha} \qquad \frac{\partial^2 U}{\partial B^2} = -\alpha(1-\alpha) A^{\alpha} B^{-\alpha-1}$$

3. The optimal allocation of resources across the two goods will have a predictable form.
4. General form: Let p_A = the price of each unit of A, and p_B = the price of each unit of B. Then solving

$$\max_{A,B} \; U(A,B) = A^\alpha B^{1-\alpha} \quad \text{s.t.} \quad w = p_A A + p_B B$$

results in

$$A^* = \frac{\alpha w}{p_A}, \; B^* = \frac{(1-\alpha)w}{p_B}$$

5. If your constrained optimization is rusty, practice deriving this result.

4.2 Economic Implications

1. *This will save you time on exams!*
2. This implies that individuals with these types of preferences devote a constant share of wealth to each good. The share of wealth devoted to each good corresponds to the exponent on that good in the utility function. In the above example, the individual spends αw on A and $(1-\alpha)w$ on B.
3. Notice, the price of B does NOT appear in the formula for the optimal amount of A. Similarly the price of A appears nowhere in the optimal amount of B.

4.3 Example 1: Homer's problem, revisited

1. Knowing what we know now, how could we have solved Homer's problem in a faster way?

Another useful short-cut: One characteristic of the optimal combination (assuming you are not at a corner) is that the marginal rate of substitution equals the price ratio (which occurs when the indifference curve is tangent to the budget constraint). *This is true in general, not just for Cobb-Douglas preferences.*

$$\frac{MU_A}{MU_B} = \frac{P_A}{P_B} \quad \Leftrightarrow \quad \frac{\partial U / \partial A}{\partial U / \partial B} = \frac{P_A}{P_B}$$

2. Solve Homer's problem using this shortcut.

Next, let Homer's utility function have a new representation: $U'(s,b) = \log(U(s,b)) = \frac{1}{2}\log s + \frac{1}{2}\log b$.

3. Is this a monotone transformation of the original utility function? Why or why not?

4. Find Homer's MRS using this representation. How does this relate to the MRS we found before? What does this imply for how much beer and pizza Homer chooses?

4.4 Example 2: Peggy Hill and labor supply

Peggy Hill is a substitute teacher. Every week she has to choose how many hours to work (L) and how many to spend on other activities (leisure). Peggy earns \$10 for every hour she works and therefore has \$10*L to spend on all other goods. (There are 168 hours in a week.) The price of all other goods has been normalized to 1. Peggy's utility is: $U(c,h) = \frac{1}{4}ch - (h-20)^2$, where h is the number of hours of leisure Peggy has.

1. Solve for c and h using $\frac{\partial U/\partial c}{\partial U/\partial h} = \frac{P_c}{P_h}$.

2. Now, Texas introduces a 20% wage tax. How much does Peggy choose to work now?